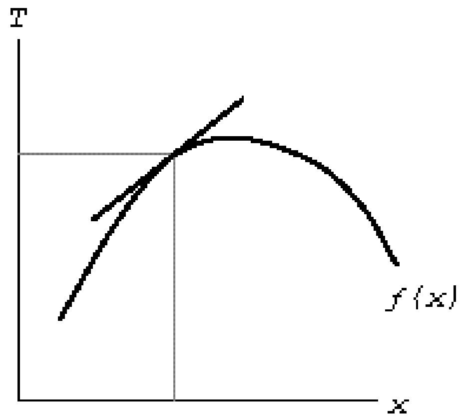
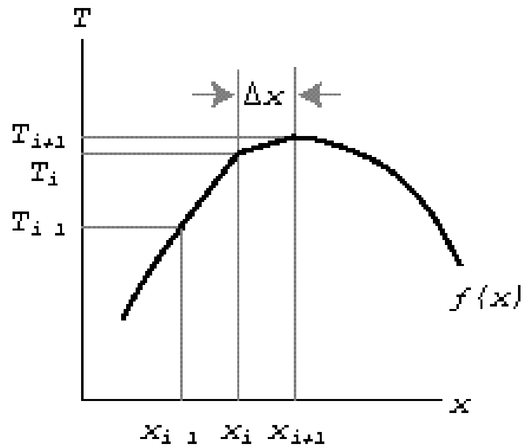
Finite Difference Method

The finite difference method is a means of solving differential equations numerically by using a finite value for Δx *rather than* $\Delta x \rightarrow 0$.





The slope *dT/dx* can be approximated at the point x_i in several ways. The *forward differencing* technique uses

$$\frac{dT}{dx} \approx \frac{T_{i+1} - T_i}{x_{i+1} - x_i} = \frac{T_{i+1} - T_i}{\Delta x}$$

The *backward differencing* technique uses

$$\frac{dT}{dx} \approx \frac{T_i - T_{i-1}}{\Delta x}$$

The *central differencing* technique uses

$$\frac{dT}{dx} \approx \frac{T_{i+1} - T_{i-1}}{2\Delta x}$$

Approximations of differential equations of order 2 or higher follow the same general scheme, using the difference between the forward and backward differences to calculate the curvature at a point:

$$\frac{d^2T}{dt^2} = \frac{d\left(\frac{dT}{dx}\right)}{dx} \approx \frac{\left(\frac{T_{i+1} - T_i}{\Delta x}\right) - \left(\frac{T_i - T_{i-1}}{\Delta x}\right)}{\Delta x} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}$$

So, this works fine for changes in space, but what about changes in *time*? For central differencing we can write

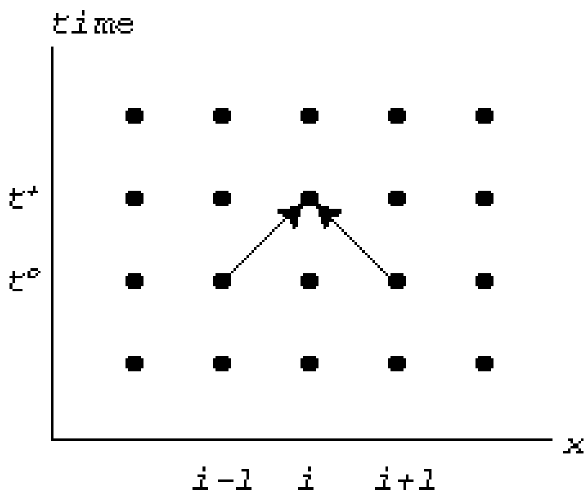
$$\frac{dT}{dt} \approx \frac{T^{t_+} - T^{t_o}}{\Delta t}$$

where T^{t_o} indicates the temperature at the present time step and T^{t_+} indicates the temperature at the next (future) time step. Thus, the diffusion equation is written

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \approx \frac{T^{t_+} - T^{t_o}}{\Delta t} = \kappa \frac{T_{i+1}^{t_o} - 2T_i^{t_o} + T_{i-1}^{t_o}}{\Delta x^2}$$

and the solution for T at the next (future) time step is

$$T_i^{t_+} = T_i^{t_o} + \frac{\kappa \Delta t}{\Delta x^2}\left(T_{i+1}^{t_o} - 2T_i^{t_o} + T_{i-1}^{t_o}\right)$$

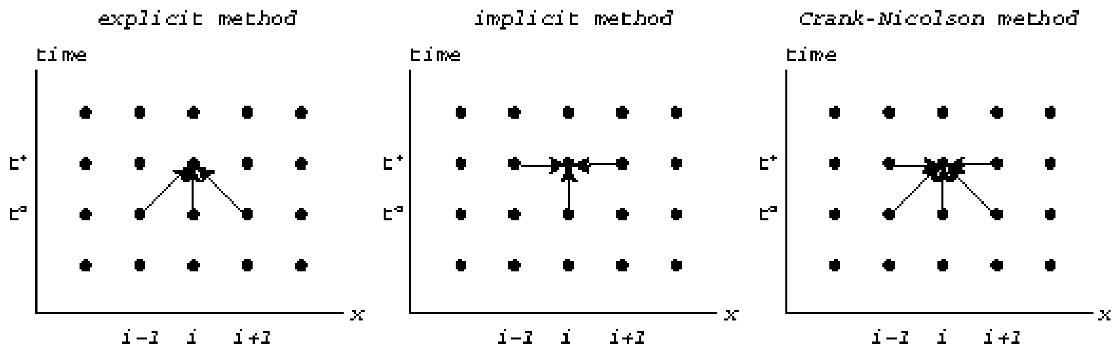


The term

$$\frac{\kappa \Delta t}{\Delta x^2}$$

in the above equation is the *Fourier cell constant* and must be smaller than 0.25 for the equation to remain stable. You must check your finite-element model for accuracy by comparing it to an analytical solution or reducing the time step and seeing that the result does not change.

There are four methods commonly used in finite-difference calculations:



The alternating-direction implicit method divides each time step in two: one half timestep that is explicit in one direction and implicit in the other and a second half timestep that is the opposite.